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Cascades and Self-Organized Criticality

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We generalize the model of self-organized critical systems to cases where due to some internal degrees of freedom the local conservation law is violated. This can be realized by taking a transfer ratio different from the critical one in a sand pile model (global violation) or allowing fluctuations around the critical ratio (local violation). In the first case the deviation from the critical ratio R is a critical parameter and the characteristic avalanche size diverges as $|R|^{-\psi}$. In the second case the global conservation assures criticality; however, our numerical results indicate that the model is in a new universality class.

KEY WORDS: Self-organized criticality; conservation; fluctuations; universality.

1. INTRODUCTION

Dissipative systems which drive themselves into a critical state are called self-organized critical systems (SOC) by Bak *et al.*[1] (BTW). For illustration they introduced a toy sand-pile model where the essential point is the local conservation law (conservation of sand grains) and the existence of a critical height difference above which the grains topple down. Avalanches generated by an external perturbation can be observed on all length scales and their size and lifetime distributions obey power laws.[1-4] Originally, the SOC were thought to be responsible also for $1/f$ or flicker noise,[1] but it was demonstrated that generally the scaling of the lifetimes and sizes does not necessarily lead to a nontrivial power spectrum.[5,6]

Experimental work related to SOC has been carried out on sand piles,[6] water drops on a window pane,[7] and magnetic domain struc-

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tures.[8] Possible relations to semiconductor physics[1] and earthquakes[10] have also been emphasized. Of course, many fractal growth phenomena[11] can also be considered as SOC, and relations to other models of statistical physics have been established in analytical calculations.[12–15]

The term self-organized criticality was invented in order to emphasize that, in contrast to usual systems where fine tuning is necessary, in the considered models and physical problems criticality is built up spontaneously. Using the sand pile language, the slope of the pile (which could play the role of the temperature[16]) is driven automatically to criticality such that avalanches on all scales appear. However, a restriction on another parameter of the problem has to be made: The perturbation (external sand flux) has to be infinitesimally small.

A crucial point in obtaining SOC behavior is that a conservation law for the fluctuating variable should hold.[1,12,17] For sand grains this is a natural assumption; however, for systems with some hidden degrees of freedom conservation could be globally or locally violated. For example, in a system where the cellular automaton equations (see below) are written down for some sort of energy variable, conservation could be violated globally by pure dissipation and locally by exchange with the host lattice. We shall mention specific examples in later parts of this paper.

Our aim here is to investigate the effect of global and local violation of conservation on the SOC behavior. In Section 2 we discuss the role of conservation in building up the SOC state and describe possible ways leading to violation of conservation. It turns out that the degree of the global violation is a critical parameter and in Section 3 we show our corresponding numerical results. In the case of the local violation of the conservation law, where, globally, on the average, conservation holds, criticality is maintained, however, with nonuniversal exponents (Section 4). We close the paper with a discussion and a summary.

2. CONSERVATION LAWS AND CASCADES

The original paper of Bak *et al.*[1] used the picture of sand piles. A discrete height variable z defined on a d-dimensional lattice describes the state of the system. The excitation is given by increments of unity of z at random positions. If a prescribed uniform threshold z_c is reached at some site, a "sliding" takes place: A part of the sand above that site is distributed among the neighbors. For the square lattice, which is our concern henceforth, we have

$$z(x, y) \Rightarrow z(x, y) - 4 \tag{1a}$$

$$z(x \pm 1, y \pm 1) \Rightarrow z(x \pm 1, y \pm 1) + 1 \tag{1b}$$

These equations express the conservation of sand grains. After a sliding, a grain may raise the sand above the threshold at a neighbor; a new sliding takes place and so on, so that avalanches are created. Due to the conservation,[12] the avalanches do not have a characteristic size: The self-organized critical state develops. The conservation is manifested in the fact that the transfer ratio r is unity, $r =$ (number of particles sliding down)/(number of particles appearing at the neighbors).

There are many areas of physics and also other fields where in such chain reactions avalanches or cascades develop. However, the conservation necessary to achieve criticality is not always assured. The transfer ratio describing the elementary events is often an externally variable parameter and/or it is a random quantity. We give a short list of examples for systems with random transfer ratio having an externally controllable mean:

(i) In nuclear reactors where the avalanches of thermal neutrons are the relevant objects, the average transfer ratio is externally controlled by the cadmium rods to keep the reactor working in the critical mode.[18]

(ii) In avalanche diodes the electrical field strength (that is, the voltage on the diode) is the external parameter which determines the average transfer ratio (so-called multiplication factor of hot electrons).[19]

(iii) In lasers the externally variable pumping energy determines the average transfer ratio.[20]

According to these considerations we generalize Eq. (1):

$$z(x, y; t+1) = z(x, y; t) - A(x, y; t) \tag{2a}$$

$$z(x \pm 1, y \pm 1; t+1) = z(x \pm 1, y \pm 1; t) + B(x, y; t) \tag{2b}$$

again for the square lattice. Now the transfer ratio is a fluctuating quantity and the average $\langle r \rangle = \langle A/4B \rangle$ can be different from unity.

For numerical inverstigations we generalized the SOC cellular automaton used by Kadanoff *et al.*[21] and Manna.[2] The system is excited at randomly chosen sites (x, y) by increments C which we took as constant in one simulation,

$$z(x, y; t+1) = z(x, y; t) + C \tag{3}$$

In practice we take always $C = \langle B \rangle$. When the $z(x, y)$ reaches an arbitrary constant threshold value z_c, then a sliding described by Eqs. (2) takes place. The mean $\langle r \rangle$ of the transition ratio $r(x, y; t)$ and its deviation R from the value belonging to the (local) conservation can be written as

$$\langle r \rangle = \langle A(x, y; t) \rangle / 4C \tag{4a}$$

$$R = 1 - \langle r \rangle \tag{4b}$$

respectively. Obviously, R is a "fine tuning parameter" corresponding to the physical situations described above.

We carried out two kinds of related numerical experiments. In the first one, to be described in the next section, we dealt with the case

$$A(x, y) = A > 4C \tag{5}$$

where the conservation is homogeneously violated in space. In the second computation we kept $R = 0$ (global conservation holds) and investigated the effect of fluctuations, i.e., of local violations of conservation. For this second case the applied relations are

$$A(x, y; t) = 4 + \xi(x, y; t) \tag{6a}$$

$$C = 1 \tag{6b}$$

where ξ is a uniformly distributed random integer number in the range $[-3, 3]$. The random number is generated at the point (x, y) whenever a sliding takes place there.

3. SUBCRITICAL AVALANCHES

First, we remark on the meaning of the distribution functions by which we described the statistics. We use the term "distribution function" $D(a)$ for the probability density of a and the term "integrated distribution function" for the integral

$$P(a) = \int_a^\infty D(a')\, da' \tag{7}$$

of the probability density. For the experimental investigation the application of the integrated distribution is often better, because it is less sensitive for the scattering of data. [Of course, if $D(a) \sim a^{-\alpha}$, then the integrated distribution is also a power function: $P(a) \sim a^{1-\alpha}$.]

In this section, we present the results of computer simulations on the cellular automaton defined by Eqs. (2)–(4) under the condition given by relation (5), i.e., we consider the case of global violation of the conservation. If the transfer ratio is larger than unity ($R < 0$), the avalanches are expected to have a characteristic lifetime after which they die out. For $R > 0$ the avalanches explode after a characteristic time. We concentrate on the first case, since it is easier to investigate.

We took an $L \times L = 256 \times 256$ square array, and measured the distribution function of the avalanche sizes. The integrated corresponding distribution functions are shown in Fig. 1 for several transition ratios R. It

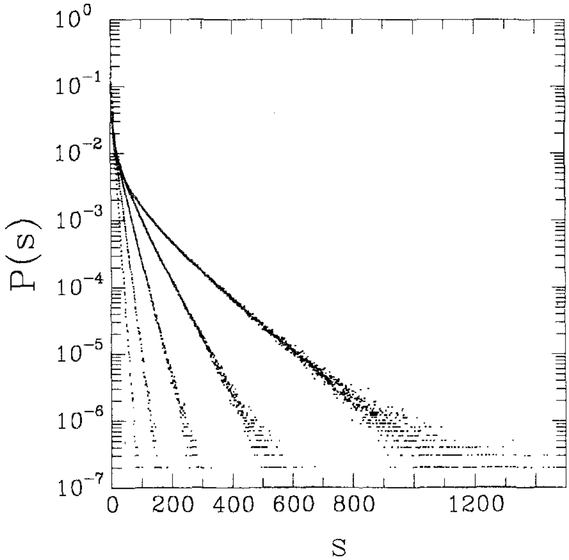

Fig. 1. Integrated avalanche size distributions in the subcritical region. $R = 0.8$, 0.4, 0.2, 0.1, and 0.05 from right to left. The distribution can be described asymptotically as $D(s) \sim \exp(s/s^*)$ with $s^* = 14.85$, 28.06, 55.10, 116.5, and 250.7, leading to $s^*(R) \propto |R|^{-0.98 \pm 0.05}$.

can be seen that in the large-s limit the distribution is an exponential function:

$$D(s) \sim \exp[-s/s^*(R)] \tag{8}$$

where s is the total number of impacts (number of particle moves) in an avalanche and $s^*(R)$ is the characteristic avalanche size. This characteristic length turns out to be related to R by the following scaling:

$$s^*(R) \sim |R|^{-\psi} \tag{9}$$

By analyzing our data we find for the exponent ψ approximately 1 (cf. Fig. 1).

This dependence of s^* on R is in full analogy with usual critical phenomena where R plays the role of the reduced temperature. The parameter R corresponds to the fine tuning parameters in the physical phenomena mentioned above which control the processes and assure criticality if it is desired. The correspondence to criticality is also expressed in the increasing fluctuations when approaching $R = 0$: this can clearly be seen in the plot in Fig. 1.

4. LOCALLY VIOLATED CONSERVATION

In the former section we showed that upon taking an R different from 0, the system is driven out of criticality. It is natural to assume that $R=0$ is enough to provide criticality irrespective of the possible local deviations from this global conservation condition. However, it is possible that the exponents characterizing the universality class of the models will be different from those obtained for global conservation.

In order to investigate the situation with locally violated conservation, we allowed fluctuations around the value $\langle A \rangle$ for down slidings [according to Eqs. (6)]. This was implemented in the following way: Whenever the critical height z_c was reached we generated an integer random number $\xi \in [-3, 3]$ and introduced the move $z \Rightarrow z-4+\xi$. The increments at the neighbors remained always equal to unity.

We have carried out simulations on $L \times L$ lattices with $L=16 \times 2^I$, $I=0, 1,..., 6$, and generated 10^5–10^6 avalanches. The larger the lattice is, the better is the critical behavior developed, and therefore the simulations become exceedingly time consuming. Accordingly, one needs large samples in order to see the power laws, but if the sample is very large, the statistics obtained in the available computer time will not be satisfactory. We have found that the results for $L=512$ were the most useful.

In Fig. 2, the integrated distribution function of avalanche sizes is shown for system size $L=512$. The integrated distribution is a power function; consequently, the distribution function $D(s)$ is also a power function:

$$D(s) \sim s^{-\tau} \tag{10}$$

However, the exponent τ has a systematic system size dependence (6%), in accordance with the simulations without fluctuations.[2] We obtained $\tau = 1.515 \pm 0.020$ from our data by a parabolic extrapolation to $L \to \infty$ in a τ vs. $1/L$ plot. We can compare our exponent τ with the value 1.22 obtained[2] for the system having no local conservation violation, $r=1$.

For the critical system one expects a power law decay also in the distribution function $D(T)$ of avalanche lifetimes. The measured integrated distribution is plotted for $L=512$ in Fig. 3; it is indeed a power function in a wide range.

Consequently, the distribution function is a power function, too:

$$D(T) \sim T^{-y} \tag{11}$$

The effective exponent y shows again a systematic system size dependence (8%). A similar extrapolation to $L \to \infty$ as described above leads to $y=$

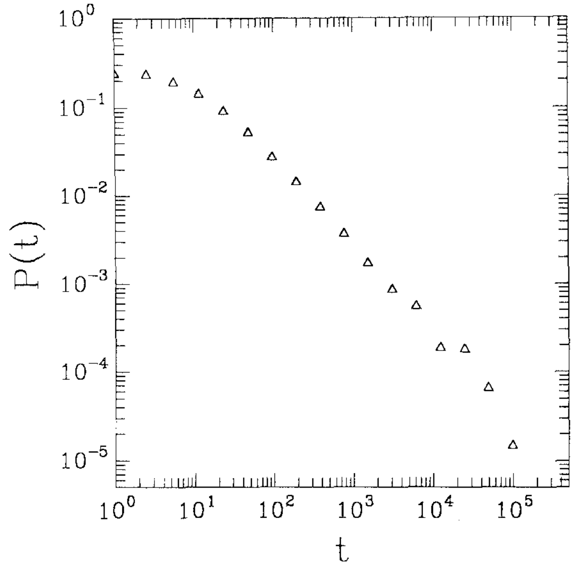

Fig. 2. Integrated avalanche size distribution at global conservation ($R=0$) but with fluctuating transfer ratios. The lattice size is $L=512$.

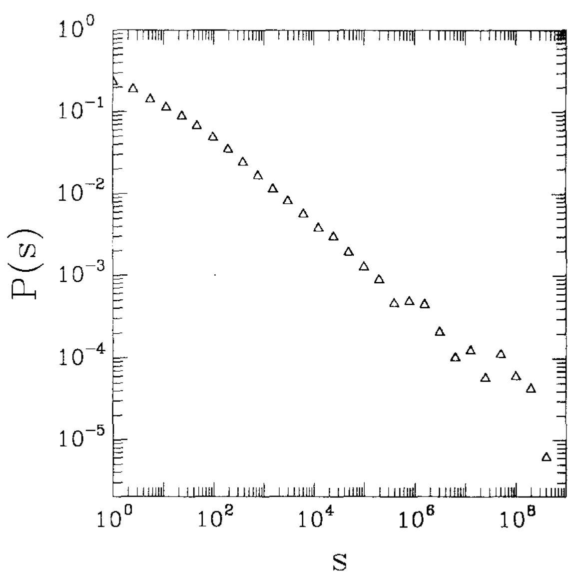

Fig. 3. Integrated avalanche lifetime distribution; $R=0$, $L=512$.

2.00 ± 0.03. This value is again significantly different from the case where no local violation of conservation is allowed[2]: $y = 1.38$.

The dependence of the avalanche lifetime T on the avalanche size s is also a power function (see Fig. 4):

$$T(s) \sim s^x \tag{12}$$

The obtained numerical value for this exponent is $x = 0.51 \pm 0.,01$, which satisfies the relation[5,1] $x = (1 - \tau)/(1 - y)$. It is interesting to note that in spite of the size-dependent effective exponents τ and y, x is rather insensitive to changes in L.

Knowing the exponents τ and x, we can calculate the power density spectrum[5] of the total avalanche currents. If $\tau + 2x < 3$, a $1/f^2$ (non-stationary) noise sets in. In our case $\tau + 2x = 2.535 \pm 0.04$, so a $1/f^2$ noise is expected. It is important to note that in the model without local violation of conservation $\tau + 2x = 2.38$, and thus our model has a somewhat "less divergent" spectrum. In Fig. 5, the measured power density spectrum for several system sizes is plotted. It can be seen that it has a $1/f^2$ dependence with a cutoff at low frequencies where the spectrum becomes white. This cutoff is due to the finite size of the system and it corresponds to the upper cutoffs of the avalanche lifetime distribution functions.

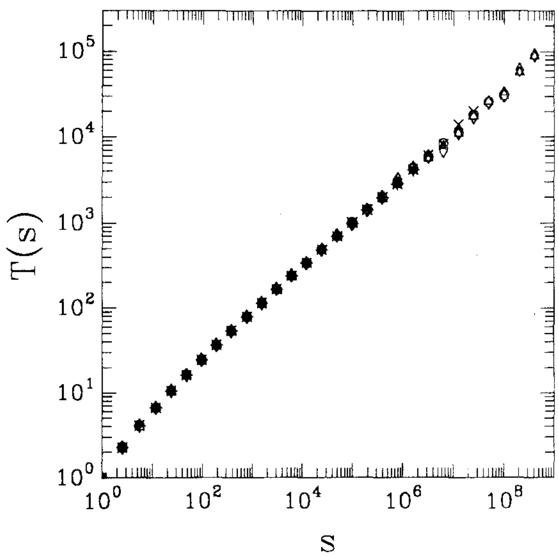

Fig. 4. Avalanche lifetime vs. size for all considered lattice size for $R = 0$. We get $x = 0.58 \pm 0.01$ for the exponent described in Eq. (12).

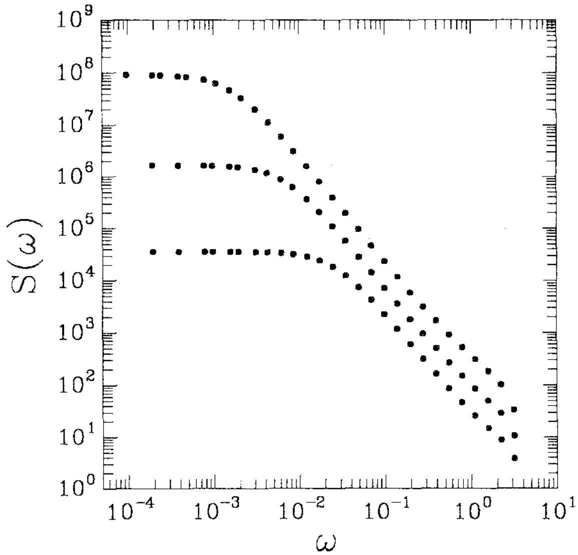

Fig. 5. The total power density spectrum of the system of avalanches; $R = 0$. It is a Lorentzian-like spectrum, dominated by the longest avalanches. $L = 16$, 32, and 64 from the bottom to the top.

5. SUMMARY AND DISCUSSION

We have demonstrated that self-organized critical systems can be considered as special cascade-forming phenomena where the transfer ratio is just critical ($R = 0$). In many physical systems R has to be fine tuned in order to achieve the desired criticality. As examples, nuclear reactors, avalanche diodes, and lasers have been mentioned. When comparing with usual critical phenomena, R plays the role of the reduced temperature or some other relevant field. Together with the external flux,[16] which has to be infinitesimally small, we have now a full two-parameter scaling framework.

We have numerically investigated the approach of criticality for $R < 1$ and obtained exponential decay in the avalanche size distribution with characteristic avalanche sizes which diverge when $r \to 1$. The parameter ψ describing this divergence can be considered as a critical exponent. An interesting and nonunderstood feature is that the average height $\langle z \rangle(R)$ does not seem to approach continuously its value for $R = 0$.

In the other case, when the global conservation was maintained ($R = 0$), but local fluctuations were allowed, we obtained exponents significantly different from the homogenous case. This is reminiscent of the situation of dynamic critical phenomena where the universality classes depend also on the conservation properties of the order parameter. Since the phenomenon we are investigating is intrinsically dynamic, all exponents seem to depend on the kind of characteristic conservation law.

Our numerically observed exponents are very close to $\psi = 1$, $\tau = 3/2$, and $y = 2$, the latter being the mean field exponents of the problem.[12,14,16] It would be interesting to see whether our system is indeed a mean field one and if it is so, why the critical dimension is shifted down at least to 2.

Even with our new exponents the inequality criterion[5] for a nontrivial noise spectrum is not fulfilled. However, it seems that the changes go in the proper direction and therefore further modifications along the given line could be promising from this point of view.

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REFERENCES

1. P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. Lett.* **59**:381 (1987); *Phys. Rev. A* **38**:364 (1988).
2. S. S. Manna, *J. Stat. Phys.* **59**:509 (1990).
3. P. Grassberger and S. S. Manna, *J. Phys.* (Paris) **51**:1077 (1990).
4. B. Grossmann, H. Guo, and M. Grant, *Phys. Rev. A* **41**:1495 (1980).
5. J. Kertész and L. B. Kiss, *J. Phys. A* **23**:L433 (1990).
6. J. H. Jensen, K. Christensen, and H. C. Fogedby, *Phys. Rev. B* **40**:7425 (1989).
7. H. M. Jaeger, Liu Chu-heng, and S. R. Nagel, *Phys. Rev. Lett.* **62**:40 (1989).
8. I. M. Jánosi and V. K. Horváth, *Phys. Rev. A* **40**:5232 (1989).
9. K. L. Babcock and R. M. Westerwelt, *Phys. Rev. Lett.* **64**:2168 (1990).
10. J. M. Carlson and J. S. Langer, *Phys. Rev. Lett.* **62**:2632 (1989).
11. T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).
12. T. Hwa and M. Kardar, *Phys. Rev. Lett.* **62**:1813 (1989).
13. Y. C. Zhang, *Phys. Rev. Lett.* **63**:470 (1989).
14. D. Dhar and R. Rhamaswamy, *Phys. Rev. Lett.* **63**:1659 (1989).
15. S. P. Obukhov, in *Random Fluctuations and Pattern Growth*, H. E. Stanley and N. Ostrowsky, eds. (Kluwer, Boston, 1989).
16. C. Tang and P. Bak, *Phys. Rev. Lett.* **60**:2347 (1988).
17. G. Grinstein, D.-H. Lee, and S. Sachdev, *Phys . Rev. Lett.* **64**:1927 (1990).
18. A. M. Weinberg and E. P. Wigner, *The Physical Theory of Nuclear Chain Reactions* (University of Chicago Press, Chicago, Illinois, 1958).
19. C. L. Hemenway, R. W. Henry, and M. Caulton, *Physical Electronics*, 2nd ed. (Wiley, New York, 1967).
20. H. Haken, *Synergetics*, 3rd ed. (Springer, Heidelberg, 1983).
21. L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, *Phys. Rev. A* **39**:6524 (1989).

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